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The field equations generated by the square of the scalar curvature: solutions of Kasner type

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Abstract. The general solution of the equation $\delta R^2/\delta g_{ij} = 0$ is found under the assumption that the metric has the generic form $ds^2 = \Sigma_{a=1}^3 e^{2\alpha_a(t)}(dx^a)^2 - e^{2\beta(t)} dt^2$. Its relation to the special solution characterised by the constancy of R is examined.

1. Introduction

In the general theory of relativity the vacuum field equations express the vanishing of the functional derivative of the scalar curvature R of the V_4 within which the formalism of the theory operates. In place of R one may contemplate other invariants $\bar{K}$ of the Riemann tensor, for if $P^{ij} := \delta\bar{K}/\delta g_{ij}$ one still has the identity $P^{ij}{}_{;j} = 0$ which plays a role in the context of conversation laws. Leaving R aside, most investigations have centred around the family of quadratic invariants $K = aR^2 + bR_{ij}R^{ij}$ $(a, b =$ constant). This is in effect the most general Lagrangian which is a quadratic form in the components of the Riemann tensor (e.g. Buchdahl 1960, § 2).

Although one may doubt whether any Lagrangian which does not reduce to R in the weak-field limit is likely to be relevant to the theory of gravitation, the study of the equations generated by the invariant K above is at least of formal interest. I showed long ago (Buchdahl 1948a, b) that they are satisfied by an arbitrary Einstein space and more recently (Buchdahl 1973) considered the question of the existence of static, regular, asymptotically flat solutions. However, very little is known about the explicit form of solutions (excluding the case where these represent an Einstein space and disregarding the case $3a + b = 0$). Even with the choice $b = 0$ and assuming the metric to be static and spherically symmetric the problem is intractable although it is reducible to the problem of finding the solution of a certain single, ordinary, non-linear second-order differential equation (Buchdahl 1962). The only tangible result in this context is that there exist no sufficiently often differentiable, asymptotically flat solutions the scalar curvature of which does not vanish everywhere.

In the light of the situation just outlined it seems desirable to investigate the possibility of so choosing K and the generic form of the metric that the general solution of the corresponding equations can be obtained explicitly. To this end I examine in this paper the equations

$$\frac{\delta R^2}{\delta g_{ij}} = 0 \tag{1.1}$$

when the metric is assumed to be of 'Kasner type', that is to say, when it has the generic form

$$ds^2 = \sum_{a=1}^{3} e^{2\alpha_a(t)}(dx^a)^2 - e^{2\beta(t)}\, dt^2. \tag{1.2}$$

Its general solution is found and its relationship to the solution characterised by the constancy of the scalar curvature is examined.

2. The field equations

The explicit form of the equations (1.1) is

$$R_{;ij} + RR_{ij} - g_{ij}(\Box R + \tfrac{1}{4}R^2) = 0. \tag{2.1}$$

From (2.1) it follows by transvection with g^{ij} that

$$\Box R = 0. \tag{2.2}$$

For the metric (1.2) the only surviving equations are as follows:

$$R[\ddot{\alpha}_a + (\dot{\theta} - \dot{\beta})\dot{\alpha}_a] + \dot{R}\dot{\alpha}_a + \tfrac{1}{4}\, e^{2\beta}R^2 = 0, \qquad (a = 1, 2, 3) \tag{2.3}$$

$$\ddot{R} - \dot{\beta}\dot{R} + (\ddot{\theta} - \dot{\theta}\dot{\beta} + \phi) + \tfrac{1}{4}\, e^{2\beta}R^2 = 0. \tag{2.4}$$

Here a dot denotes differentiation with respect to t and

$$\theta := \sum_{a=1}^{3} \alpha_a, \qquad \phi := \sum_{a=1}^{3} \dot{\alpha}_a^2. \tag{2.5}$$

(2.2) takes the explicit form

$$\ddot{R} + (\dot{\theta} - \dot{\beta})\dot{R} = 0. \tag{2.6}$$

3. Solution of the equations when $R \neq$ constant

As noted in § 1, when $R = \text{constant} =: 4\lambda \neq 0$ (2.1) reduces to

$$R_{ij} = \lambda g_{ij}, \tag{3.1}$$

i.e. the equations are satisfied by an arbitrary Einstein space; whilst when $\lambda = 0$, (2.1) is somewhat trivially satisfied when the V_4 has zero scalar curvature. Let it therefore now be assumed that R is not constant. In that case one can always so choose the time-like coordinate t that

$$R = t; \tag{3.2}$$

a device which does not affect the generic form of the metric. Then (2.6) at once shows that

$$\beta = \theta + b, \tag{3.3}$$

where b is a constant of integration. As a consequence of (3.2) and (3.3) equations (2.3) reduce to

$$t\ddot{\alpha}_a + \dot{\alpha}_a + \tfrac{1}{4}t^2\, e^{2\beta} = 0 \qquad (a = 1, 2, 3). \tag{3.4}$$

Summing over a one then obtains an equation for β, namely

$$t\ddot{\beta}+\dot{\beta}+\tfrac{3}{4}t^2\,e^{2\beta}=0.$$

This is elementary and has the solution

$$e^{-\beta}=(\sqrt{3}/4n)t^{3/2}(ct^n+c^{-1}t^{-n}), \tag{3.5}$$

where c and n are constants of integration.

Inspection of (3.4) reveals that the functions $t\ddot{\alpha}_a+\dot{\alpha}_a$ are independent of the value of a. Therefore

$$\alpha_a=\alpha_1+c_a\ln t+d_a, \qquad (a=2,3)$$

where the c_a and d_a are constants. The latter may be omitted since they may be removed by a change of scale of the coordinates x^2 and x^3. It follows that

$$\alpha_a=\tfrac{1}{3}\theta+\nu_a\ln t, \qquad (a=1,2,3) \tag{3.6}$$

where the ν_a are three constants such that

$$\sum\nu_a=0. \tag{3.7}$$

It remains to ensure that (2.4) is satisfied. The easiest way to do this is as follows. By (3.3) and (3.6)

$$\phi=\sum(\tfrac{1}{3}\dot{\beta}+\nu_a t^{-1})^2=\tfrac{1}{3}\dot{\beta}^2+t^{-2}\sum\nu_a^2.$$

Insert this in (2.4) and use (3.2) and (3.3). Upon eliminating the term $\frac{1}{4}t^2\,e^{2\beta}$ from the resulting equation one finds that $B:=e^{-\beta}$ must satisfy

$$t^2\ddot{B}-2t\dot{B}-\tfrac{3}{2}\Big(\sum\nu_a^2\Big)B=0.$$

This is compatible with (3.6) only if

$$\sum\nu_a^2=\tfrac{1}{6}(4n^2-9). \tag{3.8}$$

This result shows incidentally that the value of n cannot be less than $\frac{3}{2}$.

The constant b in (3.3) is redundant since it may be removed by a change of scale of the coordinates x^1, x^2, x^3. The solution of (2.1) thus takes the form

$$ds^2=e^{\frac{2}{3}\beta}\sum_a t^{2\nu_a}(dx^a)^2-e^{2\beta}\,dt^2, \tag{3.9}$$

where β is given by (3.6) and the ν_a are subject to the two conditions (3.7) and (3.8).

As a reliable check on these results one may calculate the right-hand side of the equation

$$R=-e^{-2\beta}(2\ddot{\beta}-\dot{\beta}^2+\phi)$$

using the expressions for β and ϕ obtained above. It turns out that $R=t$, consistently with (3.2).

4. Solution of $R_{ij}=\lambda g_{ij}$

It will shortly be useful to have the solution of (3.1) for the metric (1.2) at hand. This problem has been considered by Petrov (1964) for the more general case of a V_n of signature $2-n$ but certain conditions to be satisfied by constant parameters which occur in the metric coefficients are missing. (The original paper is not accessible to me.)

Using the previous notation, equations (3.1) read explicitly

$$\ddot{\alpha}_a + \dot{\theta}\dot{\alpha}_a = -\lambda, \qquad (a = 1, 2, 3) \tag{4.1}$$

$$\ddot{\theta} + \phi = -\lambda. \tag{4.2}$$

It will suffice to take $\lambda > 0$. Then write $3\lambda =: k^2$. Summing (4.1) over a gives

$$\ddot{\theta} + \dot{\theta}^2 + k^2 = 0,$$

whence, for a suitable choice of origin of the t-coordinate,

$$e^{\theta} = A \sin kt, \tag{4.3}$$

where A is a constant of integration. Using (4.3), (4.1) may be solved to give

$$\dot{\alpha}_a = \tfrac{1}{3}k \cot kt + k\nu_a \operatorname{cosec} kt, \tag{4.4}$$

where, on account of (4.3), the three constants of integration ν_a must satisfy the condition

$$\sum \nu_a = 0. \tag{4.5}$$

Insertion of (4.3) and (4.4) in (4.2) then shows that they must also satisfy the condition

$$\sum \nu_a^2 = \tfrac{2}{3}. \tag{4.6}$$

Integrating (4.4),

$$\alpha_a = \tfrac{1}{3} \ln \sin kt + \nu_a \ln \tan(\tfrac{1}{2}kt),$$

the new constants of integration here being omitted since they can be accounted for by a suitable change of scale of the coordinates x^a. Thus finally

$$ds^2 = \sin^{2/3}(kt) \sum_a \tan^{2\nu_a}(\tfrac{1}{2}kt)(dx^a)^2 - dt^2, \tag{4.7}$$

where the ν_a must satisfy the conditions (4.5) and (4.6), these being the conditions referred to at the beginning of this section.

Now write $\nu_a =: n_a - \frac{1}{3}$ and set

$$\tau := 2k^{-1} \tan(\tfrac{1}{2}kt). \tag{4.8}$$

Then, with a suitable change of scale of the x^a, (4.7) can be written

$$ds^2 = \cos^{2/3}(\tfrac{1}{2}kt) \sum_a \tau^{2n_a}(dx^a)^2 - dt^2, \tag{4.9}$$

where

$$\sum n_a = 1, \qquad \sum n_a^2 = 1. \tag{4.10}$$

When $\lambda \to 0$, i.e. $k \to 0$, this reduces at once to the original Kasner metric (see Petrov 1964)

$$ds^2 = \sum_a t^{2n_a}(dx^a)^2 - dt^2. \tag{4.11}$$

5. Remark on the relation between the cases $R \neq$ constant and $R=$ constant

The following procedure is instructive even if it bears a somewhat heuristic character. In (3.5), (3.9) set

$$c =: (3/4k^2)^n =: (4\lambda)^{-n} \qquad \text{and} \qquad \nu_a =: n\nu'_a.$$

Then make the transformation of coordinates

$$t = (c^{-1}\tan\tfrac{1}{2}kt')^{1/n}, \qquad x^a = h_a x'^a, \quad \text{(not summed)}$$

where $h_a = (3/4n^2)^{1/6}(3/4k^2)^{\nu_a - 1/2}$. In the first place we now have

$$R = 4\lambda(\tan\tfrac{1}{2}kt')^{1/n}. \tag{5.1}$$

Furthermore, the metric (3.9) becomes

$$ds^2 = \sin^{2/3}kt' \sum_a \tan^{2\nu'_a - 1/n}(\tfrac{1}{2}kt')(dx'^a)^2 - dt'^2, \tag{5.2}$$

where

$$\sum \nu'_a = 0, \qquad \sum \nu'^2_a = \tfrac{2}{3} - \tfrac{3}{2}n^{-2}. \tag{5.3}$$

In the limit $n \to \infty$ (5.2) and (5.3) reduce exactly to the corresponding equations (4.7), (4.5) and (4.6) obtained in § 4, whilst (5.1) consistently reduces to $R = 4\lambda$. Bearing in mind that the solution with $\lambda = 0$ can in turn be obtained from (4.7) as shown previously, one sees that (3.9) is in effect the most general solution of the equation (1.1) when the metric is prescribed to have the generic form (1.2).

6. The special case $n = \frac{3}{2}$

The metric (3.9) will be spherically symmetric provided $\nu_1 = \nu_2 = \nu_3$. On account of (3.7) and (3.9) this is possible only if $\nu_a = 0$ and therefore $n = \frac{3}{2}$. In this case, upon making suitable changes of scale of the coordinates and bearing in mind that the equations (1.1) are insensitive to the provision of ds^2 with a constant factor, the solution of these equations may be exhibited in the form

$$ds^2 = (t^3+1)^{-2/3}(dx^2 + dy^2 + dz^2) - (t^3+1)^{-2}\,dt^2; \tag{6.1}$$

and $R = 12t$. This is conformally flat, by inspection; but (3.9) is not conformally flat for any other values of the ν_a.

7. Modified Kasner type metric

In place of (1.2) one may also consider in the present context a 'modified Kasner type metric', i.e. one which differs from the former only in that the α_a and β are taken to be functions of x^1 rather than of t. Not surprisingly the corresponding solution bears a strong formal resemblance to (3.9):

$$ds^2 = e^{2\beta}(dx^1)^2 + e^{\frac{2}{3}\beta}[(x^1)^{2\nu_2}(dx^2)^2 + (x^1)^{2\nu_3}(dx^3)^2 - (x^1)^{2\nu_1}\,dt^2] \tag{7.1}$$

with

$$e^{-\beta} = (\sqrt{3}/4n)(x^1)^{3/2}[c(x^1)^n - c^{-1}(x^1)^{-n}],$$

where c, n, ν_1, ν_2, ν_3 are constants of integration the last three of which are again subject to (3.7) and (3.8).

8. Concluding remark

In (7.1) the constants ν_a are naturally real and therefore the least allowed value of n is $\frac{3}{2}$. However, one can contemplate also complex values of the ν_a provided that at the same time one makes an appropriate complex transformation of coordinates. The resulting metric will still satisfy (1.1). The following is a simple example. Set $\nu_1 = a + ib$, $\nu_2 = a - ib$, a, b real; and make the transformation

$$x^1 = x, \qquad y = (iu + v)/\sqrt{2}, \qquad x^3 = z, \qquad t = (u + iv)/\sqrt{2}.$$

Then (7.1) becomes

$$ds^2 = e^{2\beta}\,dx^2 + e^{\frac{2}{3}\beta}\{x^{2a}[\cos(2b \ln x)(dv^2 - du^2) + 2\sin(2b \ln x)\,du\,dv] + x^{2\nu_3}\,dz^2\} \tag{8.1}$$

which has again the correct signature +2. (3.7) and (3.8) become

$$2a + \nu_3 = 0, \qquad 3a^2 - b^2 = \tfrac{1}{3}(n^2 - \tfrac{9}{4}) \tag{8.2}$$

so that one no longer has the necessary restriction $n \geqslant \frac{3}{2}$. Although this is an interesting conclusion it is not strictly relevant to the present context since a metric of Kasner type, as defined initially, must be diagonal; which (8.1) is not. (8.1) is merely one example of the more general class of 'one-dimensional' metrics of signature 2. Those which are Ricci-flat have been determined by Dautcourt *et al* (1962) and to find all those which satisfy (1.1) would require an analogous investigation.

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References

Buchdahl H A 1948a *Proc. Edinburgh Math. Soc.* **8** 89–94
—— 1948b *Proc. Natn. Acad. Sci. USA* **34** 66–8
—— 1960 *J. Math. Phys.* **1** 537–41
—— 1962 *Nuovo Cim.* **23** 141–57
—— 1973 *Proc. Camb. Phil. Soc.* **74** 145–8
Dautcourt G, Papapetrou A and Treder H 1962 *Ann. Phys., Lpz.* **9** 330–6
Petrov A S 1964 *Einstein-Räume* (Berlin: Akademie-Verlag) chap. 2, § 14